



# An Algorithm for the Class of Pure Implicational Formulas

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## Abstract

Heusch introduced the notion of pure implicational formulas. He showed that the falsifiability problem for pure implicational formulas with  $k$  negations is solvable in time  $O(n^k)$ . Such falsifiability results are easily transformed to satisfiability results on CNF formulas.

We show that the falsifiability problem for pure implicational formulas is solvable in time  $O(k^k n^2)$ , which is polynomial for a fixed  $k$ . Thus this problem is fixed-parameter tractable.

## 1 Introduction

Since Cook's [1] proof of NP-completeness for 3-SAT, numerous studies of complexity of testing satisfiability and/or falsifiability for particular classes of propositional formulas have been done. Two important classes are CNF and DNF formulas. For CNF formulas, Cook showed that testing satisfiability is NP-complete, whereas testing falsifiability can be done in linear time. For DNF the reverse is true: testing satisfiability (resp. falsifiability) for DNF formulas can easily be reduced to testing falsifiability (resp. satisfiability) for CNF formulas. Other classes of formulas, such as Horn, extended Horn, q-Horn, and SLUR, have been shown to be solved in polynomial time (see [9] for algorithms and credits). Recursively defined hierarchies of incrementally harder classes have also been defined and studied [3, 5, 7]. Formulas on level  $k$  of these hierarchies typically can be solved in  $O(n^k)$  time.

Downey and Fellows [4] considered hierarchies of problems. Levels of such a hierarchy are distinguished by a parameter  $k$  which in some sense measures the density of an instance; any algorithm to solve instances of length  $n$  for fixed  $k$  takes time bounded by some  $c_k(1+n^{e_k})$ . Now as  $k$  increases, does the value of  $e_k$  increase, or does it hold steady while only the value of  $c_k$  increases? An algorithm of the latter type they call *fixed parameter tractable*. Fixed parameter tractability has obvious implications for algorithm efficiency on instances where  $n$  is much larger than  $k$ ; for more information about fixed parameter tractability we refer the reader to Downey and Fellows' papers. In this paper we show that the falsifiability problem, described below, for pure implicational

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formulas of length  $n$  containing at most  $k$  occurrences of  $\mathbf{f}$ , is fixed parameter tractable. In the cases of [3, 5, 7], it is not known whether there are polynomial-time fixed parameter algorithms. As far as we know, the results of this paper are the first fixed parameter tractability results for hierarchies of satisfiability.

Following the work of Peter Heusch [6], we consider formulas in a different “normal form”: formulas built up from the *proposition letters* (or *propositional variables*) with  $\rightarrow$  (implication) and a propositional *constant*  $\mathbf{f}$  for false (so  $\neg\mathbf{f}$  is always true) — plus parentheses, of course. Call a formula built up with only these connectives *pure implicational*. We consider algorithms for determining satisfiability and falsifiability of formulas containing at most two occurrences of each proposition letter and at most  $k$  occurrences of  $\mathbf{f}$ .

Actually, Heusch considered only falsifiability, and he did not allow occurrences of  $\mathbf{f}$ . Rather, he considered formulas whose only propositional connective is  $\rightarrow$ , which contain at most  $k$  occurrences of some distinguished proposition letter  $z$ , and which contain at most two occurrences of every other proposition letter. However, he showed that determining falsifiability was hardest in the case of formulas of the form

$$\alpha_1 \rightarrow (\alpha_2 \rightarrow (\dots \rightarrow (\alpha_h \rightarrow z))).$$

In this case  $z$  is set to false in every falsifying truth assignment. Thus it is equally difficult to determine falsifiability of the formula

$$\alpha_1 \rightarrow (\alpha_2 \rightarrow (\dots \rightarrow (\alpha_h \rightarrow \mathbf{f}))).$$

In this way Heusch essentially reduced his context to the one we use.

Since it slightly simplifies our arguments, we shall focus on satisfiability. It is equally difficult to test for falsifiability and to test for satisfiability, since a formula  $\phi$  is satisfiable if and only if  $(\phi \rightarrow \mathbf{f})$  is falsifiable, and  $\phi$  is falsifiable if and only if  $(\phi \rightarrow \mathbf{f})$  is satisfiable. Without loss of generality, we limit the study to satisfiability of a single pure implicational formula  $\phi$ . For suppose  $\alpha_1, \dots, \alpha_m$  are pure implicational formulas. Then  $\{\alpha_1, \dots, \alpha_m\}$  is satisfiable if and only if

$$\alpha_1 \rightarrow (\dots \rightarrow (\alpha_m \rightarrow \mathbf{f}))$$

is falsifiable.

**Theorem 1.1 (Heusch [6])** *Testing falsifiability for pure implicational formulas containing at most two occurrences of each proposition letter is NP-complete.*  $\square$

**Corollary 1.2** *Testing satisfiability for pure implicational formulas containing at most two occurrences of each proposition letter is NP-complete.*  $\square$

But by fixing the value of the extra parameter, the number of occurrences of  $\mathbf{f}$  (or the number of occurrences of  $z$  in his case), Heusch was able to get a type of tractability result for special cases. From this point on, we use  $n$  to represent the length of a pure implicational formula.

**Theorem 1.3 (Heusch [6])** *Testing falsifiability for pure implicational formulas, of length  $n$ , containing at most two occurrences of each proposition letter and at most  $k$  occurrences of  $\mathbf{f}$  can be performed in time  $O(n^k)$ .*  $\square$

The following corollary is trivial; we assume that more careful analysis of Heusch's proof would easily change the  $k + 1$  to a  $k$  below.

**Corollary 1.4** *Testing satisfiability for pure implicational formulas, of length  $n$ , containing at most two occurrences of each proposition letter and at most  $k$  occurrences of  $\mathbf{f}$  can be performed in time  $O(n^{k+1})$ .*  $\square$

The organization of the paper is as follows. In Section 2 we show how to transform pure implicational formulas to a forest of trees containing no more nodes than the number of implication connectives. The trees are such that there is a collection of "consistent paths," from root to leaf, one for each tree, if and only if the original formula is satisfiable. The search for such a collection is simplified by the fact that inconsistencies arise only between a leaf and its "shadow." In Section 3 we devise a  $O(k^k n^2)$  algorithm for searching a forest for inconsistent paths. The collection of algorithms used in all stages is presented as a whole at the end of the section.

## 2 Exploring possibilities

Our algorithm consists of three stages. The first consists of parsing and translating the given formula into a different propositional form. The second constructs at most  $k$  trees of literals, where  $k$  is the number of occurrences of  $\mathbf{f}$  in the given formula. The third stage consists of solving a combinatorial problem on the trees. The solution to this tree problem determines the satisfiability of the given formula.

### 2.1 Initial parsing/translation

The first stage of the algorithm involves replacing a pure implicational formula  $\phi$  with a set  $\Theta = \{\theta_1, \dots, \theta_m\}$  ( $m \leq n$ ) of formulas in a larger set of proposition letters, where  $\phi$  is satisfiable if and only if  $\Theta$  is. I.e.,  $\phi$  and  $\Theta$  are *equisatisfiable*.

1. We start with a pure implicational formula, for example,

$$((((a_0 \rightarrow a_1) \rightarrow \mathbf{f}) \rightarrow a_2) \rightarrow \mathbf{f}) \rightarrow ((\mathbf{f} \rightarrow a_0) \rightarrow (a_2 \rightarrow \mathbf{f}))).$$

We parse, counting levels of nesting to the *left hand side* of the  $\rightarrow$ 's; below we show nesting on the left by dropping down to lower lines:

$$\begin{array}{c} ( \\ ( \\ ( \\ ( \\ (a_0 \rightarrow a_1) \end{array} \begin{array}{c} \Downarrow \\ \Downarrow \\ \Downarrow \\ \Downarrow \end{array} \begin{array}{c} \rightarrow \mathbf{f}) \\ \rightarrow a_2) \\ \rightarrow \mathbf{f}) \\ \end{array} \rightarrow ( \begin{array}{c} \Downarrow \\ \Downarrow \end{array} \begin{array}{c} \rightarrow (a_2 \rightarrow \mathbf{f})) \end{array} \begin{array}{c} \Downarrow \\ \Downarrow \end{array} \rightarrow (a_2 \rightarrow \mathbf{f}))).$$

2. Next, at even numbered levels ( $> 0$ ) of nesting on the left, we first replace the embedded formulas above with new proposition letters.

$$\begin{array}{c} ( \\ ( \\ ( \\ ( \\ (a_0 \rightarrow a_1) \end{array} \begin{array}{c} \Downarrow \\ \Downarrow \\ \Downarrow \\ \Downarrow \end{array} \begin{array}{c} \rightarrow \mathbf{f}) \\ \rightarrow a_2) \\ \rightarrow \mathbf{f}) \\ \end{array} \rightarrow ( \begin{array}{c} \Downarrow \\ \Downarrow \end{array} \begin{array}{c} \rightarrow (a_2 \rightarrow \mathbf{f})) \end{array} \begin{array}{c} \Downarrow \\ \Downarrow \end{array} \rightarrow (a_2 \rightarrow \mathbf{f}))).$$

In order to make sure that our new set of formulas and the original formula  $\phi$  are equisatisfiable, we need also to relate  $c_i$ 's to the formulas they replace. It is natural to add in an axiom  $c_1 \leftrightarrow (a_0 \rightarrow a_1)$ . That would, however, force  $c_1$  to occur more than twice in our final translation. Fortunately, we need assert only the  $\rightarrow$  direction of the equivalence to guarantee equisatisfiability. Thus we get our translation, in this case a set  $\Theta$  of three formulas:

$$\begin{array}{c} ( \\ ( \\ ( \\ ( \\ (a_0 \rightarrow a_1) \end{array} \begin{array}{c} \Downarrow \\ \Downarrow \\ \Downarrow \\ \Downarrow \end{array} \begin{array}{c} \rightarrow \mathbf{f}) \\ \rightarrow a_2) \\ \rightarrow \mathbf{f}) \\ \end{array} \rightarrow ( \begin{array}{c} \Downarrow \\ \Downarrow \end{array} \begin{array}{c} \rightarrow (a_2 \rightarrow \mathbf{f})) \end{array} \begin{array}{c} \Downarrow \\ \Downarrow \end{array} \rightarrow (a_2 \rightarrow \mathbf{f}))). \\ c_0 \rightarrow ( \\ c_1 \rightarrow (a_0 \rightarrow a_1) \end{array}$$

We sketch the proof of equisatisfiability: If the original formula  $\phi$  has a model  $\nu$ , we expand  $\nu$  by setting  $\nu(c_i)$  to the value under  $\nu$  of the corresponding subformula of  $\phi$ , which yields a model of  $\Theta$ . Conversely, suppose  $\nu$  is a model of  $\Theta$ ; we show that  $\nu$  is also a model of  $\phi$ . As noted in our discussion of our intuition above, if for each  $c_i$ ,  $\nu(c_i)$  is the value of the formula it replaces in the expansion above — i.e., if  $\nu(c_0) = \nu((c_1 \rightarrow \mathbf{f}) \rightarrow a_2)$  and  $\nu(c_1) = \nu(a_0 \rightarrow a_1)$ , then  $\nu$  is a model of  $\phi$ . So suppose not. Pick all the highest level formulas for which this

property fails. For example, suppose here that  $\nu(c_0) \neq \nu((c_1 \rightarrow \mathbf{f}) \rightarrow a_2)$ . Since  $\Theta$  contains the formula  $c_0 \rightarrow ((c_1 \rightarrow \mathbf{f}) \rightarrow a_2)$ , it must be that  $\nu(c_0) = \text{false}$  and  $\nu((c_1 \rightarrow \mathbf{f}) \rightarrow a_2) = \text{true}$ . From the truth table for propositions nested an even number of times on the left of an  $\rightarrow$ , we can see that if the top formula is satisfied when  $\nu(c_0)$  is false, it is satisfied with *any* formula substituted in for  $c_0$ .

Note that now not only the original proposition letters (the  $a_i$ 's), but also the new ones (the  $c_i$ 's), occur at most twice in  $\Theta$ .

3. We now have three formulas in our set  $\Theta$ , each with at most one level of nesting on the left-hand side of the  $\rightarrow$ 's.

$$\begin{aligned}\theta_0 &= (c_0 \rightarrow \mathbf{f}) \rightarrow ((\mathbf{f} \rightarrow a_0) \rightarrow (a_2 \rightarrow \mathbf{f})) \\ \theta_1 &= c_0 \rightarrow ((c_1 \rightarrow \mathbf{f}) \rightarrow a_2) \\ \theta_2 &= c_1 \rightarrow (a_0 \rightarrow a_1)\end{aligned}$$

We finish with a simple substitution of tautologically equivalent formulas. We replace the *nested-on-the-left* occurrences of  $\rightarrow$  with their DNF equivalents, yielding

$$\begin{aligned}\theta_0 &= (\neg c_0 \vee \mathbf{f}) \rightarrow ((\neg \mathbf{f} \vee a_0) \rightarrow (a_2 \rightarrow \mathbf{f})) \\ \theta_1 &= c_0 \rightarrow ((\neg c_1 \vee \mathbf{f}) \rightarrow a_2) \\ \theta_2 &= c_1 \rightarrow (a_0 \rightarrow a_1);\end{aligned}$$

we replace each remaining  $\beta_1 \rightarrow (\beta_2 \rightarrow (\dots \beta_j))$  with the equivalent  $\neg \beta_j \rightarrow (\neg \beta_1 \vee \neg \beta_2 \vee \dots \neg \beta_{j-1})$ , yielding

$$\begin{aligned}\theta_0 &= \neg \mathbf{f} \rightarrow \neg(\neg c_0 \vee \mathbf{f}) \vee \neg(\neg \mathbf{f} \vee a_0) \vee \neg a_2 \\ \theta_1 &= \neg a_2 \rightarrow (\neg c_0 \vee \neg(\neg c_1 \vee \mathbf{f})) \\ \theta_2 &= \neg a_1 \rightarrow \neg c_1 \vee \neg a_0;\end{aligned}$$

and we simplify using deMorgan's laws and double negation, yielding finally

$$\begin{aligned}\theta_0 &= \neg \mathbf{f} \rightarrow (c_0 \wedge \neg \mathbf{f}) \vee (\mathbf{f} \wedge \neg a_0) \vee \neg a_2 \\ \theta_1 &= \neg a_2 \rightarrow (\neg c_0 \vee (c_1 \wedge \neg \mathbf{f})) \\ \theta_2 &= \neg a_1 \rightarrow \neg c_1 \vee \neg a_0.\end{aligned}$$

**Comment 2.1** Note that any atom that occurs only positively can be set to **true**, without affecting the satisfiability of these formulas. Therefore, before we continue, we perform this simplification. (This simplifies later exposition, and can be done in time  $O(n^2)$ .)

Note the important properties of  $\Theta$ :

- $\Theta$  and  $\phi$  are equisatisfiable.

- Each proposition letter —  $a_i$  or  $c_i$  — occurs at most twice in  $\Theta$ .
- The constant  $\mathbf{f}$  occurs  $k$  times in  $\Theta$ .
- Each  $\theta_i$  is an implication, whose hypothesis consists of a negative literal (either a  $\neg a_i$  or a  $\neg c_i$  or a  $\neg \mathbf{f}$ ), and whose conclusion consists of a disjunction, where each disjunct is the conjunct of one negative literal and zero or more (zero or one in our example) positive literals.

All occurrences of  $\mathbf{f}$  on the right-hand sides of the  $\theta_i$ 's could be simplified out, although our algorithm assumes that we do not do so. The complexity of the algorithm will be given in terms of the number  $k' \leq k$  of occurrences of  $\neg \mathbf{f}$  on the *left hand sides* of the  $\theta_i$ 's.

## 2.2 Building trees of choices

We now have a collection  $\Theta = \{\theta_1, \dots, \theta_m\}$  of formulas as described above. As we noted earlier, our strategy is now dual to the strategy of finding whether a set of Horn clauses is satisfiable. There one starts with a default truth assignment of false to every variable and uses the implications to identify variables which must be true. Here we start with a default truth assignment of true to every variable and use the implications to identify choices of variables to set to false to satisfy the formulas. If there are no occurrences of  $\neg \mathbf{f}$  in the hypotheses of the  $\theta_i$ 's (i.e.,  $k' = 0$ ) then  $\{\theta_1, \dots, \theta_m\}$  is satisfied by the all-true assignment.

We want to search for a minimal change to the all-true truth assignment to satisfy  $\{\theta_1, \dots, \theta_m\}$ . To start out, we must satisfy each of the  $k'$  formulas with head  $\neg \mathbf{f}$ . In order to do that we must satisfy one of the disjuncts of each body. Suppose we satisfy a disjunct  $\neg v$ , and suppose some other  $\theta_i$  is  $\neg v \rightarrow \neg w \vee \neg x$ ; then we will also have to set either  $w$  or  $x$  to false. To keep track of our choices, for each such formula  $\theta_i = \neg \mathbf{f} \rightarrow \dots$ , we build a tree  $T_i$  with root labeled  $\neg \mathbf{f}$ . Associated with each interior node and leaf of these trees will be a set of labels, corresponding to the literals in the relevant disjunction of some formula, as described in the algorithm below. Later we shall find an interpretation by choosing a branch through the tree; the literals labeling a node will be literals we shall be forced to satisfy when the algorithm chooses that branch.

For notational convenience below, we assume that we have moved the  $\theta_i$ 's with hypothesis  $\neg \mathbf{f}$  to  $\theta_1, \dots, \theta_{k'}$ .

### Algorithm 2.1

For  $i = 1, \dots, k'$

Build tree  $T_i$  as follows:

Create a root  $r_i$  labeled  $\neg \mathbf{f}$ .

For each disjunct  $l_1 \wedge l_2 \wedge \dots \wedge l_j$  of the consequent of  $\theta_i$ ,

Create a child of  $r_i$  with labels  $l_1, l_2, \dots, l_j$ .  
 Mark that child unexplored  
 While there are unexplored nodes:  
   Pick any unexplored node  $w$  and mark it explored.  
   If the node has a negative literal  $\neg a$  as a label  
     and does not have  $\mathbf{f}$  as a label  
     If there is a  $\theta_j$  with hypothesis  $\neg a$   
       (there can be only one such  $\theta_j$  by the 2-occurrence restriction)  
       For each disjunct  $l_1 \wedge l_2 \wedge \dots \wedge l_j$  of the consequent of  $\theta_j$   
         Create a child of  $w$  with labels  $l_1, l_2, \dots, l_j$   
         Mark that child unexplored.  
 Remove all nodes all of whose descendants are labeled  $\mathbf{f}$ .  
 If a node does not have any children, mark it as a leaf.

As we build the trees, we also build the following:

- An array Occurrence, indexed by the proposition letters, whose index- $a$  position contains pointers to the (at most 2) occurrences of  $a$  in the  $T_i$ 's.
- For each node  $w$ , an array, indexed by the proposition letters, whose index- $a$  position stores whether  $a$  and/or  $\neg a$  label nodes on the path to  $w$  from its tree's root.

**Comment 2.2** We immediately use the auxiliary data structures for a simplification: if any negative literal  $\neg a$  occurs negatively in two leaves, those leaves can, without loss of generality, be removed, and their parent nodes marked as leaves.

It is easy to see how to construct the  $\Theta$ 's and the  $T_i$ 's in time  $n^2$ . (That is almost certainly not the optimal time. However, other phases of our satisfiability algorithm are also  $O(n^2)$ , so this suffices for our final analysis.)

We shall use the trees above to help us search for a satisfying truth assignment for  $\{\theta_1, \dots, \theta_m\}$ . We shall build up this assignment in stages, combining *partial* truth assignments. To simplify the exposition, we introduce a notion of a *set of literals* satisfying such a formula. We shall consider a *truth assignment* to be a set  $\mathcal{A}$  of literals such that, for each proposition letter  $a$ , exactly one of  $a$  and  $\neg a$  is in  $\mathcal{A}$ .

**Definition 2.1** Let  $IMPL$  be the set of formulas whose only propositional connectives are  $\mathbf{f}$ ,  $\neg$ ,  $\rightarrow$ ,  $\vee$ , and  $\wedge$ . For a set of literals,  $\mathcal{A}$ , we say that  $\mathcal{A}$  satisfies  $\delta$ , written  $\mathcal{A} \models \delta$ , for  $\delta \in IMPL$ , if the following hold.

- $\mathcal{A} \models \neg f; \mathcal{A} \not\models f$ .
- For  $l$  either  $a$  or  $\neg a$ ,  $a$  a proposition letter,  $\mathcal{A} \models l$  if  $l \in \mathcal{A}$ .
- $\mathcal{A} \models \alpha \vee \beta$  if  $\mathcal{A} \models \alpha$  or  $\mathcal{A} \models \beta$ .
- $\mathcal{A} \models \alpha \wedge \beta$  if  $\mathcal{A} \models \alpha$  and  $\mathcal{A} \models \beta$ .
- $\mathcal{A} \models \alpha \rightarrow \beta$  if  $\mathcal{A} \not\models \alpha$  or  $\mathcal{A} \models \beta$ .

**Definition 2.2** Let  $T_i$  be one of the trees above and let  $P$  be any path on  $T_i$  (from the root to any leaf of  $T_i$ ). Then  $\mathcal{A}_P$  is the set of labels of nodes of  $P$ .

So, to satisfy  $\{\theta_1, \dots, \theta_m\}$  with a *partial truth assignment* we have an easy solution: First pick one path  $P_i$  from each  $T_i$  ( $1 \leq i \leq k'$ ), and start with the interpretation  $\mathcal{A}_0 = \mathcal{A}_{P_1} \cup \dots \cup \mathcal{A}_{P_{k'}}$ . That is enough to satisfy chains of inferences starting with  $\theta_1, \dots, \theta_k$ . Then set  $\mathcal{A} = \mathcal{A}_0 \cup \{\text{proposition letters } v : v, \neg v \notin \mathcal{A}_0\}$ . This (vacuously) satisfies all the  $\theta_i$ 's not addressed before, since each  $\theta_i$  that is not used in some  $T_i$  has a negated literal as hypothesis. If  $\theta_i$  has hypothesis  $\neg a$ , and  $\theta_i$  was not used in the construction of any  $T_j$ ,  $\neg a$  does not occur as a label in any of the  $T_j$ 's. Thus,  $\neg a$  does not occur in  $\mathcal{A}_0$ , so either  $a \in \mathcal{A}_0$  or  $a \in \mathcal{A}$ .

The only difficulty with the above construction is that  $\mathcal{A}_0$  may well be inconsistent, i.e., it may contain both  $a$  and  $\neg a$  for some proposition letter  $a$ . Thus we have:

**Theorem 2.1** Let  $\theta_1, \dots, \theta_m$  and  $T_1, \dots, T_{k'}$  be as above, and let  $\mathcal{A}$  be a truth assignment. Then:

1. If  $\mathcal{A} \models \{\theta_1, \dots, \theta_m\}$  then there are paths  $P_i$  of  $T_i$ ,  $i = 1, \dots, k'$  such that  $\mathcal{A}_{P_i} \subseteq \mathcal{A}$ .
2. If (1) there are paths  $P_i$  of  $T_i$ ,  $i = 1, \dots, k'$  such that  $\mathcal{A}_{P_i} \subseteq \mathcal{A}$ , and (2) if, for each proposition letter  $a$  not appearing in any of the  $P_i$ 's,  $a \in \mathcal{A}$ , then  $\mathcal{A} \models \{\theta_1, \dots, \theta_m\}$ .
3.  $\{\theta_1, \dots, \theta_m\}$  is satisfiable if and only if there are paths  $P_i$  of  $T_i$ ,  $i = 1, \dots, k'$  such that for no atom  $a$  are both  $a$  and  $\neg a$  in  $\mathcal{A}_{P_1} \cup \mathcal{A}_{P_2} \cup \dots \cup \mathcal{A}_{P_{k'}}$ .  $\square$

**Remark 2.2** The above theorem can be simplified further. Since each variable can appear at most twice in  $\{\theta_1, \dots, \theta_m\}$ , and since a variable is used to label a node only when it appears as a consequent in one of the  $\theta_i$ 's, the only way we can have both  $a, \neg a$  in  $\mathcal{A}_{P_1} \cup \mathcal{A}_{P_2} \cup \dots \cup \mathcal{A}_{P_{k'}}$  is for  $a$  to label some node of some  $P_i$  and for  $\neg a$  to label the leaf of some  $P_j$ .

Note that, by the 2-occurrence property for atoms, and by the construction of the  $T_i$ 's, a negative literal appears in these trees either uniquely in an interior node of some  $T_i$ , or in some leaf (or possibly two leaves).



If we pick two paths, they are inconsistent with each other iff there is an atom  $a$  such that  $a$  appears in one path, and  $\neg a$  appears in the other. If this happens, then  $\neg a$  is a label of some leaf  $w$ . Therefore, if we wish to avoid inconsistencies, once we pick a leaf,  $w$ , with a label  $\neg a$ , we must avoid all paths through the (possibly nonexistent) node  $y$  with label  $a$ . In other words, no paths can contain  $y$ , nor any of its descendants. We call the set of nodes in the subtree (cone) rooted at  $y$  the *shadow* of  $w$  ( $Shadow(w)$ ).

If a leaf does not have a negative label, we say it has an *empty shadow*.

**Remark 2.3** Let  $\{\theta_1, \dots, \theta_m\}$  and  $T_1, \dots, T_{k'}$  be as above. Then  $\{\theta_1, \dots, \theta_m\}$  is satisfiable iff there is a set of leaves  $w_1 \in T_1, \dots, w_{k'} \in T_{k'}$  where no  $w_i$  is in any  $w_j$ 's shadow.

**Remark 2.4** Using the data structures specified in Algorithm 2.1, given any two leaves  $w_i, w_j$ , we can test in constant time whether  $w_i \in Shadow(w_j)$ .

## 2.3 The reduction to graph theory

In the previous subsections we reduced, in at most quadratic time, satisfiability of the original pure implicational formula to the following combinatorial problem on trees, where  $n$  is the length of the original formula  $\phi$  (including parentheses — this allows for our new constants  $c_{\gamma_i}$ ) and  $k'$  is less than or equal to the number of occurrences of  $\mathbf{f}$  in  $\phi$ .

Remember the following facts:

- $T_1, \dots, T_{k'}$  are labeled trees, with a total of at most  $n$  nodes.
- There is a partial function  $Shadow$  from leaves to sets of nodes. For each leaf  $w$ , if  $Shadow(w) \neq \emptyset$ ,  $Shadow(w)$  consists of just the descendants of a single node in just one of the trees.

### Definition 2.3

- A shadow-independent set in  $\{T_1, \dots, T_{k'}\}$  is a set of leaves  $\{w_i \in T_i : 1 \leq i \leq k'\}$  such that no leaf  $w_i$  is in any  $w_j$ 's shadow (including its own).
- The shadow problem for  $\{T_1, \dots, T_{k'}\}$  is, “Does  $\{T_1, \dots, T_{k'}\}$  have a shadow-independent set?”

**Proposition 2.5** If  $S_1, S_2$  are two shadows, then one of the following must hold: (1)  $S_1, S_2$  are disjoint, (2)  $S_1 \subseteq S_2$ , or (3)  $S_2 \subseteq S_1$ .

A simple upper bound on the difficulty of the shadow problem can be found by reducing it to the independent set problem. (Since this reduction gives only an upper bound on the complexity of the problem, it is omitted.) Thus we can apply to the shadow problem any general algorithm for finding whether a graph has a  $k$ -independent set. The fastest known (at least to us) algorithm is an  $O(n^{k(2+\epsilon)/3})$  algorithm by Nešetřil and Poljak [8], where  $2 + \epsilon$  is the best known exponent for fast matrix multiplication (see [2]). This is an improvement on Heusch's algorithm, but it is still not fixed parameter tractable.

Thus, short of showing that the  $k$ -independent set problem is fixed parameter tractable, our approach must amount, essentially, to finding special features of the resultant graph which allow for faster algorithms. We already have such a result in Proposition 2.5. In the remaining sections we show that the special features given in Proposition 2.5 are in fact strong enough: we give an  $O(n^2)$  algorithm (for each  $k$ ) for this problem.

### 3 Solving the Shadow Problem

#### 3.1 Easy Simplifications

Before we start, we identify, for each leaf  $w$  of each tree  $T_i$ , the tree  $T_j$  in which  $Shadow(w)$  lies. If  $w$  has an empty shadow, we pick an arbitrary tree  $T_j \neq T_i$  and declare that  $w$  has an empty shadow in tree  $T_j$ .

Given the set of trees  $\{T_1, \dots, T_k\}$ , we can perform the following simplifications in quadratic time. Clearly, none of the simplifications changes the answer to the shadow problem. Note that the simplification steps must be repeated until no further changes are made or unfalsifiability has been determined:

**Algorithm 3.1** *Repeat the following simplifications until there are no more simplifications to be performed.*

1. *If any leaf shadows itself, delete the leaf — since no unshadowed set could contain such a leaf.*
2. *If any leaf's shadow is in its own tree, but does not include the leaf itself, delete the shadow (i.e., remove the label  $\neg a$  from the leaf and the label  $a$  from the base of the shadow) — since we need to pick only one leaf from each tree anyway.*

*Redefine the leaf's shadow to be an empty shadow in some other tree.*

3. *If any leaf's shadow is an entire tree  $T_i$ , remove the leaf — since no unshadowed set could contain such a leaf.*

4. Remove any node which is not marked as a leaf, yet has no children. (This just prunes interior nodes whose children have all been removed.)
5. If any node has only one child, merge it with its child, labeling the merged node with all literals labeling either the original node or its child. If its child was a leaf, mark it as a leaf.
6. If any tree becomes empty, return "unsatisfiable".

**Remark 3.1** Algorithm 3.1 can be performed in time  $O(n^2)$ .

### 3.2 A partition into shadow patterns

To find a shadow-independent set of leaves, we must pick one leaf from each tree. Think of a directed edge going from the leaf's tree to the shadow's tree (which, by the simplifications above, must be a different tree). Call the resultant graph on the  $k$  trees the *shadow pattern* of that set of leaves. Note that each node in a shadow pattern has out-degree one, so there are  $(k-1)^k$  possible shadow patterns on the  $k$  trees. We shall partition our algorithm by looking separately at each of the  $(k-1)^k$  possible shadow patterns, checking to find if there is an unshadowed set inducing that shadow pattern.

Say that a leaf  $l_i$  *matches a shadow pattern*  $P$  if  $l_i$  is in tree  $T_i$ , the edge in the shadow pattern from  $T_i$  goes to  $T_j$ , and  $l_i$ 's shadow is in  $T_j$ . A set of leaves *matches a shadow pattern* if every leaf in the set matches the pattern.

Note that, to search for a shadow-independent set matching a shadow pattern  $P_h$ , we can clearly search each weakly connected component of  $P_h$  (i.e., each component of the undirected version of  $P_h$ ) separately. The following algorithm is then clearly correct, since each choice of  $k$  leaves, one from each tree, matches one of the shadow patterns. We use it as the outside control of our solution algorithm.

#### Algorithm 3.2

For each shadow pattern  $P_h$ ,  $1 \leq h \leq (k-1)^k$ ,  
  Make a copy  $T_i^h$  of each of the original trees  $T_i$   
  Delete from  $T_i^h$  all leaves which do not match shadow pattern  $P_h$ .  
  Simplify  $T_i^h$  as in Algorithm 3.1.  
  For each weakly connected component  $W$  of  $P_h$   
    search for a shadow-independent set on  $W$ . (†)  
  If every weakly connected component has a shadow-independent set  
    return "satisfiable."  
Return "unsatisfiable".

The difficulty, of course, is in step (†). We shall show that we can implement that step in  $O(n_W^2)$ , where  $n_W$  is the total number of nodes in the trees  $T_i^h \in W$ . It will follow that the above algorithm can be performed in time  $O(k^k n^2)$ , since we have already noted that Algorithm 3.1 takes only quadratic time. (Clearly, it suffices to call Algorithm 3.1 on each component  $W$  separately.)

### 3.3 Solving for a fixed component of a fixed shadow pattern

All that is left is to show, as promised above, that we can test whether a weakly connected component  $\{T_{w_1}^h, \dots, T_{w_g}^h\}$  of  $\{T_1^h, \dots, T_k^h\}$  has a shadow-independent set in time quadratic in the number of nodes in the component. The essential intuition used by the algorithm is the following:

1. In acyclic parts of the component, we can easily perform the test by brute force, working our way in from the leaves. We do this by pruning away nodes which *cannot* be in a shadow-independent set.
2. In the cyclic part of the graph, we show that, unless the pattern of shadows (after the pruning described above) has a specific, easily recognized structure described in Theorem 3.3, there is a shadow-independent set.

Since  $P_h$  is a directed graph on  $k$  nodes where each vertex has out-degree one, each of its weakly-connected components is a directed tree with edges directed toward the root plus one back edge. Or, another way of looking at it, each component consists of a single directed cycle  $c$  plus a set of mutually disjoint trees  $\{t_1, \dots, t_r\}$  (directed from leaves to root), where the root of each tree is also a member of the cycle, and otherwise the cycle and the trees are disjoint. We start with a separate pruning algorithm for the trees  $t_i$ :

#### Algorithm 3.3

##### Pruning Algorithm:

**Input:** A component  $W$  of  $P_h$  along with the member trees  $T_i^h$ , consisting of:

1. A collection of mutually disjoint trees,  $t_1, \dots, t_f$ , and
2. A single cycle  $c$ , where
  - a. the roots of the  $t_g$ 's are elements of  $c$ ,
  - b. no other nodes are shared by the  $t_g$ s and  $c$ .

The nodes of  $c$  and the  $t_g$ 's are all trees  $T_i^h$  created before.

##### Action:

From each tree, prune those nodes that cannot be in any shadow-independent set.

Identify the  $c$  and the  $t_g$ 's.

For each tree  $t_g$   
     For each node  $T_i^h$  of  $t_g$ , starting with the leaves of  $t_g$   
         and working backwards to the root  
         If  $T_i^h$  has children in  $t_g$   
             For each child  $T_j^h$  of  $T_i^h$  in  $t_g$   
                 For each leaf  $w$  of  $T_j^h$   
                     If  $w$  is in the shadow of every leaf of  $T_j^h$   
                         delete  $w$  from  $T_j^h$ .  
 Simplify  $W$  as in Algorithm 3.1.  
 If the simplification algorithm returned "unsatisfiable"  
     return that there is no shadow-independent set matching  $P_h$ .

### Lemma 3.2

1. If a leaf  $l_i$  of a tree  $T_i^h$  is deleted by Algorithm 3.3, that leaf cannot be included in any shadow-independent set from the input trees matching the given shadow pattern.
2. If the pruning algorithm returns that there is no shadow-independent set matching  $P_h$  (last line of the algorithm), then that result is correct.
3. Every shadow-independent set on the cycle  $c$ , after the pruning algorithm is finished, can be expanded to a shadow-independent set on all of the component  $W$ .
4. The total running time of Algorithm 3.3 is  $O(|W|^2)$ , where by  $|W|$  we mean the total number of nodes in all the member trees  $T_i^h$  in  $W$ .

### Proof:

1. Straightforward.
2. If the simplification algorithm reports unsatisfiability, it is because one of its trees has become empty. If any tree becomes empty, there are no leaves left to put into the shadow-independent set.
3. Suppose we have a shadow-independent set  $I$  on the pruned cycle  $c$ . Remember that some elements of  $c$  are roots of trees  $t_g$  and thus have been pruned. We must expand  $I$  to a shadow-independent set on all of  $W$ . Remember also that every tree has its root in  $c$ . Working out from the roots of the trees  $t_g$  to the leaves of the  $t_g$ 's, we have by construction, that no matter what leaf  $l$  we picked for the independent set in the parent node  $T_i^h$  in some  $t_g$ , we can pick leaves  $l'$  in each of the children  $T_j^h$  (children in  $t_g$ ) which do not shadow  $l$ . Pick any such  $l'$ 's and continue recursively.

4. The cycle  $c$  and the trees  $t_g$  can be identified by a standard depth-first search. Since the search is just on  $W$ , the time is  $O(|W|)$ . The remainder of the algorithm is basically a matter of checking through all pairs of leaves from adjacent trees, hence the quadratic bound.  $\square$

**Definition 3.1** *By a leaf-descendent of a node  $x$  of a tree, we mean any descendent of  $x$  which happens to be a leaf, or  $x$  itself if that is a leaf.*

**Theorem 3.3** *Let  $c$  be a simple cycle in  $P_h$  as above, containing trees  $T_0^h, \dots, T_{k''-1}^h$ . (We simply renumber the trees to get them numbered as stated.) There is no shadow-independent set of leaves from  $T_0^h, \dots, T_{k''-1}^h$  matching  $P_h$  if and only if the following condition holds:*

**Condition (†):**

*After Algorithm 3.1 is called (again) on  $T_0^h \cup \dots \cup T_{k''-1}^h$ , either some  $T_i^h$  becomes empty or*

1. *the root of each of these  $T_i^h$ 's has exactly two children — call them  $c_i^{\text{left}}$  and  $c_i^{\text{right}}$  (in some order);*
2. *if we choose which child of the root of each  $T_i^h$  to name  $c_i^{\text{left}}$  and which  $c_i^{\text{right}}$  correctly, every leaf-descendent of  $c_i^{\text{left}}$  (resp.,  $c_i^{\text{right}}$ ) has the same shadow, which is*  
*for  $i < k'' - 1$ :  $c_{(i+1)}^{\text{right}}$  (resp.  $c_{(i+1)}^{\text{left}}$ ) and all of its descendants, or*  
*for  $i = k'' - 1$ :  $c_0^{\text{left}}$  (resp.  $c_0^{\text{right}}$ ) and all of its descendants.*

**Proof:**

$\Rightarrow$ : We are given the cycle  $c$  consisting of  $T_0^h, \dots, T_{k''-1}^h$ . By the simplifications (Algorithm 3.1) we performed, each  $T_i^h$  either (i) consists of just one node, which is not in the shadow of any other node in  $T_0^h, \dots, T_{k''-1}^h$ , or (ii) has at least two children, and no node in  $c$  shadows more than one of these children.

We start by looking at a very gross approximation to the existence of a shadow-independent set. For each  $T_i^h$  consisting of more than a single node (and hence having at least two children of the root by our simplifications), let  $M_i^0, \dots, M_i^{c_i}$  be the maximal subtrees of  $T_i^h$ , i.e. the subtrees starting at the children of the root. For the single node  $T_i^h$ 's, let  $M_i^0 = T_i^h$ . Now we build a directed *accessibility graph*  $\mathcal{G}$  on these  $M_i^j$ 's. One can think of  $\mathcal{G}$  as the cycle  $c$  with most of its nodes — those  $T_i^j$ 's which are nontrivial — split into pieces, one per subtree at the root of  $T_i^j$ .

The nodes of  $\mathcal{G}$  are the  $M_i^j$ 's. There is a directed edge in  $\mathcal{G}$  from  $M_i^j$  to  $M_{(i+1) \bmod(k'')}^{j'}$  if there is a leaf of  $M_i^j$  whose shadow is *disjoint* from  $M_{(i+1) \bmod(k'')}^{j'}$ . Since no leaf of  $M_i^j$  shadows

more than one of the  $M_{(i+1)\bmod(k'')}^{j'}$ 's, there are edges from each  $M_i^j$  to at least all but one of the  $M_{(i+1)\bmod(k'')}^{j'}$ 's, and if there is only one  $M_{(i+1)\bmod(k'')}^{j'}$ , then there is an edge from  $M_i^j$  to  $M_{(i+1)\bmod(k'')}^{j'}$ . Thus also there is at least one edge out of each  $M_i^j$ .

Now we claim that if  $\mathcal{G}$  has a  $k''$ -cycle then  $\{T_0^h, \dots, T_{k''-1}^h\}$  has a shadow independent set. For suppose  $M_0^{j_0}, M_1^{j_1}, \dots, M_{k''-1}^{j_{k''-1}}$  is a  $k''$ -cycle. By the definition of  $\mathcal{G}$ , there is a leaf  $l_i$  in each  $M_i^{j_i}$  whose shadow is disjoint from  $M_{(i+1)\bmod(k'')}^{j_{i+1}}$ . Pick any such sequence of leaves  $l_0, l_1, \dots, l_{k''-1}$ . Clearly,  $\{l_0, l_1, \dots, l_{k''-1}\}$  is shadow-independent.

The basic strategy for constructing a shadow-independent set from  $\mathcal{G}$  is this: out of each node  $M_i^j$  there is at least one edge in  $\mathcal{G}$ . So do a depth first search of  $\mathcal{G}$  to find out what nodes of  $\mathcal{G}$  are accessible from each  $M_0^j$  in  $e$  edges,  $1 \leq e \leq k''$ . (This search is in the correctness proof; our algorithm does *not* need to perform the search since it does not need to construct the satisfying assignment: it needs only to determine whether one exists.) It follows from the above that, from each  $M_0^j$ , (1) some node  $M_0^{j'}$  is accessible in  $k''$  edges, and (2) all but at most one of the  $M_0^{j'}$ 's are accessible in  $k''$  steps.

We first show that if  $\mathcal{G}$  has no  $k''$ -cycles then the root of each  $T_i^h$  has exactly two children. First suppose the root of some  $T_i^h$  has only one child; without loss of generality, we may assume it to be  $T_0^h$ . Since we performed Algorithm 3.1 immediately before entering this phase,  $T_0^h$  has only one node, and  $M_0^0 = T_0^h$ . Tracing forward  $k''$  steps along any path through  $\mathcal{G}$ , we come back to some  $M_0^j$  — and thus to  $M_0^0$ , since that's the only one there is. This gives a directed  $k''$  cycle, and thus an unshadowed set.

Next, suppose that there is no shadow-independent set, and every  $T_i^j$ 's root has at least two children, but that some  $T_i^j$  has more than 2 children. Without loss of generality, we may assume that  $T_{k''-1}^j$ 's root has at least 3 children. Consider the nodes in  $\mathcal{G}$  accessible in  $k'' - 1$  and in  $k''$  edges from  $M_0^0$  and  $M_0^1$ . At least two of  $M_{k''-1}^0$ ,  $M_{k''-1}^1$ , and  $M_{k''-1}^2$  are accessible from each of  $M_0^0$  and  $M_0^1$ ; thus one of them,  $M_{k''-1}^j$ , is accessible from both. Now there must be an edge from  $M_{k''-1}^j$  to at least one of  $M_0^0$  and  $M_0^1$ ; this edge will complete a  $k''$  cycle. That completes showing property (†1).

Next we show part of property (†2): assuming that there is no shadow-independent set, out of each  $M_i^j$  there is exactly one edge in  $\mathcal{G}$ . Without loss of generality, choose  $M_0^{left} = M_0^0$  and  $M_0^{right} = M_0^1$ . We know there is a path of length  $k''$  out of  $M_0^{left}$ ; since we assumed there is no shadow-independent set in the  $T_i^h$ 's, and hence  $k''$ -cycle in  $\mathcal{G}$ , every such path must lead to  $M_0^{right}$ . Similarly, there is a  $k''$ -length path out of  $M_0^{right}$ , and every such path must lead to  $M_0^{left}$ . Finally, if there were more than one  $k''$ -length path out of either — say out of  $M_0^{left}$  — then one of those paths would have to share a node with the path out of  $M_0^{right}$ , which would imply the existence of a  $k''$ -cycle. (Thus we see that  $\mathcal{G}$  is a simple  $2k''$  cycle, doubled

up on  $c$  like a Möbius strip.) Without loss of generality, we may call the nodes of  $\mathcal{G}$  accessible from  $M_0^{left}$  (resp.,  $M_0^{right}$ ) in  $i$  edges,  $1 \leq i \leq k'' - 1$ ,  $M_i^{left}$  (resp.,  $M_i^{right}$ ), and we have that the only edge out of  $M_{k''-1}^{left}$  (resp.,  $M_{k''-1}^{right}$ ) goes to  $M_0^{right}$  (resp.,  $M_0^{left}$ ).

Now we prove the rest of property (†2). Suppose that there is some  $i$  where there is a leaf of  $M_i^{left}$  whose shadow is not all of  $M_{i+1}^{right}$ . (The case for some leaf of  $M_i^{right}$  whose shadow is not all of  $M_{i+1}^{left}$  is analogous.) Without loss of generality, we may assume that  $i = 0$ . We shall now construct a  $k''$ -cycle in  $\mathcal{G}$ , contradicting the assumption that there is no shadow-independent set in the set of  $T_i^h$ 's.

Pick  $w_0$  to be a leaf in  $M_0^{left}$  whose shadow is not all of  $M_1^{right}$ , and pick  $w_1$  to be any leaf of  $M_1^{right}$  which is not in the shadow of  $w_0$ . For  $i = 2, \dots, k''$ , pick  $w_i$  to be any leaf of  $M_i^{right}$ . By our choice of  $w_0, w_1$ ,  $w_0$  does not shadow  $w_1$ . By our analysis above,  $w_i$  does not shadow  $w_{i+1}$  for any  $i < k'' - 1$ . Finally, again by our analysis above, since  $w_{k''-1} \in M_{k''-1}^{right}$ , the shadow of  $w_{k''-1}$  is in  $M_0^{right}$ , so  $w_{k''-1}$  does not shadow  $w_0$ . So we have constructed a  $k''$ -cycle, contradicting our assumption.

⇐: Obvious. □

**Proposition 3.4** *Condition (†) of Theorem 3.3 can be checked in time  $O(|W|)$ . ( $|W|$  is defined in Lemma 3.2.)*

Assembling all the pieces we have constructed we have a faster algorithm for checking satisfiability for our class of pure implicational formulas, giving our fixed parameter tractability result.

### Algorithm 3.4 The Combined Algorithm

**Input:** *Pure implicational formula  $\phi$  with each proposition letter occurring at most twice*

**Output:** *“Satisfiable” or “Unsatisfiable”*

*Translate  $\phi$  to equisatisfiable formulas  $\{\theta_1, \dots, \theta_{k'}\}$ , as in Section 2.*

*For  $i = 1, \dots, k'$*

*Build tree  $T_i$  as follows:*

*Create a root  $r_i$  labeled  $\neg f$ .*

*For each disjunct  $l_1 \wedge l_2 \wedge \dots \wedge l_j$  of the consequent of  $\theta_i$ ,*

*Create a child of  $r_i$  labeled  $l_1, l_2, \dots, l_j$ .*

*Mark that child unexplored*

*While there are unexplored leaves:*

*Pick any unexplored leaf  $w$  and mark it explored.*

*If the leaf has an negative literal  $\neg a$  as a label*



If there is a  $\theta_j$  with hypothesis  $\neg a$   
 (there can be only one such  $\theta_j$  by the 2-occurrence restriction)  
 For each disjunct  $l_1 \wedge l_2 \wedge \dots \wedge l_j$  of the consequent of  $\theta_j$   
 Create a child of  $w$  labeled  $l_1 \wedge l_2 \wedge \dots \wedge l_j$   
 Mark that child unexplored.

For each shadow pattern  $P_h$ ,  $1 \leq h \leq (k-1)^k$ ,  
 Make a copy  $T_i^h$  of each of the original trees  $T_i$   
 Delete from  $T_i^h$  all leaves which do not match shadow pattern  $P_h$ .  
 Simplify  $T_i^h$  as in Algorithm 3.1.  
 If the simplification algorithm returned “unsatisfiable”  
     continue to next  $P_h$

Putting together the previous complexity results, we have that:

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## 4 Boolean functions representable by pure implicational formulas

It is well known that there are Boolean functions which cannot be represented by pure implicational formulas, while any such functions can be represented by implicational formulas which allow for occurrences of the Boolean constant **f**. We address the question of characterizing those Boolean functions which can be represented by pure implicational formulas and show that the number of such functions is surprisingly large.

A Boolean function  $\varphi : B^n \rightarrow B$  is called an *implicant* of the Boolean function  $\psi$  iff  $\varphi \rightarrow \psi$  is a tautology.

**Theorem 4.1** *A Boolean function  $\psi : B^n \rightarrow B$  is representable by a pure implicational formula  $\phi$  iff there is a Boolean variable  $z$ , which is an implicant of  $\psi$*

**Proof:**

Suppose  $\psi$  is represented by the pure implicational formula  $\phi$ . Then  $\phi$  has the form  $\phi_1 \rightarrow (\phi_2 \rightarrow \dots (\phi_h \rightarrow z) \dots)$ , where  $z$  is the rightmost variable of  $\phi$ .

It is easy to see that assigning  $z$  to true results in the value of true for  $\phi$ . That is,  $z$  is an implicant of  $\phi$  and so of  $\psi$ .

Now suppose the variable  $z$  is an implicant of  $\psi$ . That is,  $z \rightarrow \psi$  is a tautology. Denote by  $\psi_{z=\mathbf{f}} : B^{n-1} \rightarrow B$  the Boolean function defined by  $\psi$  when assigning  $z$  to false. Define  $\psi_{z=\mathbf{t}}$  the same way, assigning  $z$  to true. Applying Shannon decomposition to  $\psi$  and  $z$  we obtain:

$$\begin{aligned}
 \psi &= (z \wedge \psi_{z=\mathbf{t}}) \vee (\bar{z} \wedge \psi_{z=\mathbf{f}}) \\
 &= z \vee \psi_{z=\mathbf{f}} && \text{(because } z \text{ is an implicant of } \psi) \\
 &= (\psi_{z=\mathbf{f}} \rightarrow z) \rightarrow z \\
 &= (\phi_1 \rightarrow z) \rightarrow z && (\phi_1 \text{ is a Boolean formula representing } \psi_{z=\mathbf{f}} \\
 &&& \text{consisting of Boolean variables, the constant } \mathbf{f} \text{ and the operator } \rightarrow.) \\
 &= (\phi_2 \rightarrow z) \rightarrow z && (\phi_2 \text{ is obtained from } \phi_1 \text{ by substituting every} \\
 &&& \text{occurrence of } \mathbf{f} \text{ in } \phi_1 \text{ by the variable } z. \text{ That is, } \phi_2 \text{ is a pure implicational formula.)} \\
 &= \phi,
 \end{aligned}$$

and  $\phi$  is a pure implicational formula. □

Our proof of Theorem 4.1. tells us more about Boolean functions representable by pure implicational formulas. If  $\psi$  is representable by a pure implicational formula then there is a Boolean variable  $z$ , which is an implicant of  $\psi$ . If we fix however the value of  $z$  to **false**, the function  $\psi_{z=f} : B^{n-1} \rightarrow B$  can be an arbitrary Boolean function without any restriction. This immediately yields upper and lower bounds on the number  $i_n$  of Boolean functions  $\psi : B^n \rightarrow B$  representable by pure implicational formulas.

**Corollary 4.2**  $2^{2^{n-1}} \leq i_n \leq n2^{2^{n-1}}.$  □

From Corollary 4.2, the number of Boolean functions in  $n$  arguments representable by pure implicational formulas is about the square root of the number of all Boolean function in  $n$  arguments.

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